JOURNAL OF APPROXIMATION THEORY 37, 182–195 (1983)

Positive and Minimal Projections in Function Spaces

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Communicated by E. W. Cheney

Received March 2, 1982

1. INTRODUCTION

If L is a linear lattice and E is a linear subspace of L, it is natural to ask whether there is a positive projection of L onto E (a projection P is *positive*, or *monotone*, if $x \geqslant 0$ implies $Px \geqslant 0$). This is always the case, for example, when L is $L^p(\mu)$ and E is a closed linear sublattice [4, Chap. 3]. However, much less is known about the situation when L is the function space $C(X)$ (X compact, Hausdorff) with supremum norm, though for certain subspaces Korovkin's theorem implies that there is no positive projection.

In Section 2, we give necessary and sufficient conditions for there to be a positive projection of a normed linear lattice L onto an n-dimensional subspace L_n. As a corollary, we see that if M is a closed sublattice of a Banach lattice L and there is a positive projection of M onto L_n, then there is a positive projection of L onto L_n. In particular, every finite-dimensional sublattice of L admits a positive projection. When L is $C(X)$, our characterization reduces to the following: L_n admits a positive projection if and only if there exist positive functions $b_1,..., b_n$ in L_n and points $x_1,..., x_n$ of X such that $b_i(x_j) = \delta_{ij}$.

In Section 3, we study the companion problem for finite-codimensional subspaces of $C(X)$. We prove, in fact, that if X has no isolated points, then such subspaces never admit positive projections.

* Supported by the Science and Engineering Research Council of Great Britain while on leave at the University of Lancaster.

0021-9045/83/020182-14$03.00/0

In Section 4, we are concerned with projections of $C(X \times Y)$ onto certain natural subspaces. Here we consider minimal as well as positive projections. More precisely, let M be the subspace consisting of all functions of the form $\phi(x, y) = f(x) + g(y)$. Also, for fixed $x^* \in X$, $y^* \in Y$, let $C_0(X \times Y)$ be the set of functions in $C(X \times Y)$ that vanish at (x^*, y^*), and let $M_0 = M \cap C_0(X \times Y)$. Then there exists no positive projection of $C(X \times Y)$ onto M, and exactly one positive projection P^* of $C_0(X \times Y)$ onto M_0. Furthermore, if X and Y are infinite, then for any projection P of $C(X \times Y)$ onto M, $\|P\| \geqslant 3$, while for any projection P of $C_0(X \times Y)$ onto M_0, $\|P\| \geqslant \|P^*\| = 2$. These results easily generalize to the product of k spaces X^i. Also, the method of proof establishes exact estimates (in the first case) for the norms when some or all of the spaces X^i are finite.

2. Finite-Dimensional Subspaces

Our first result applies to general normed linear lattices. A *linear lattice* (or *Riesz space*) is a linear space (over the real field) with a lattice ordering $\geqslant$ such that

$$x \geqslant 0, \quad y \geqslant 0 \quad \text{implies} \quad x + y \geqslant 0,$$
$$x \geqslant 0, \quad \lambda \in \mathbb{R}^+ \quad \text{implies} \quad \lambda x \geqslant 0.$$

We use the usual notation: $\sup\{x, y\} = x \vee y$, $\inf\{x, y\} = x \wedge y$, $|x| = x \vee (-x)$. A *normed linear lattice* is a normed linear space equipped with a lattice ordering such that $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$. If the space is also complete with respect to the norm, it is called a *Banach lattice.*

Let L_n denote an n-dimensional linear subspace of a normed linear lattice L. Suppose that there is a positive projection P of L onto L_n. It is then elementary (and well-known) that the ordering of L_n is a lattice ordering: in fact, we have

$$\sup_{L_n}(x, y) = P(x \vee y)$$

for $x, y \in L_n$. (This does not mean that L_n is a sublattice of L since $x \vee y$ need not belong to L_n). The positive cone in L_n is closed, hence Archimedean. By a standard result on finite-dimensional linear lattices [4, p. 70], L_n has a basis $\{b_1, ..., b_n\}$ such that $\sum_1^n \lambda_i b_i \geqslant 0$ if and only if $\lambda_i \geqslant 0$ for all i. We deduce the following characterization of finite-dimensional subspaces that admit positive projections:

THEOREM 1. *Let L_n be an n-dimensional subspace of a normed linear lattice L. Then the following statements are equivalent*:

(i) *there is a positive projection of* L *onto* L_n,

(ii) *there exist positive elements* b_i *of* L_n *and positive linear functionals* f_i *on* L $(i = 1,..., n)$ *such that* $f_i(b_j) = \delta_{ij}$.

Proof. If (ii) holds, then a positive projection P is simply given by $Px = \sum_1^n f_i(x)\, b_i$.

Conversely, suppose that there is a positive projection P. Then L_n has a basis $\{b_1,..., b_n\}$ as above. For $x = \sum_1^n \lambda_i b_i \in L_n$, let $g_i(x) = \lambda_i$. The g_i are positive linear functionals defined on L_n, and $g_i(b_j) = \delta_{ij}$. The required functionals on L are given by $f_i = g_i \circ P$. ∎

COROLLARY. *Let* L *be a Banach lattice,* M *a closed linear sublattice. Let* L_n *be an n-dimensional subspace of* M. *If there is a positive projection of* M *onto* L_n, *then there is a positive projection of* L *onto* L_n.

Proof. Theorem 1 gives us positive linear functionals f_i defined on M. It is well-known that every positive functional on a Banach lattice is continuous ([4, p. 84]; if there were positive elements x_k with $\|x_k\| \leqslant 2^{-k}$ and $f(x_k) \geqslant k$, then no definition would be possible for $f(\sum x_k)$). Further, every continuous positive functional defined on M has a positive extension defined on L [4, p. 86]. ∎

In particular, every finite-dimensional linear sublattice admits a positive projection. As we shall see, this is far from being the case for infinite-dimensional sublattices of $C(X)$, though it is true in $L^p(\mu)$, $1 \leqslant p < \infty$ [4, p. 212].

Remark. It is sufficient in Theorem 1 if L, instead of having a lattice ordering, has an Archimedean ordering satisfying the Riesz decomposition property, that is, if $x_1, x_2 \geqslant 0$ and $0 \leqslant y \leqslant x_1 + x_2$, then $y = y_1 + y_2$, where $0 \leqslant y_i \leqslant x_i$, $i = 1, 2$. Finite-dimensional spaces with this property are order-isomorphic to $\mathbb{R}^n$ with the usual order.

The next result shows that in the case $L = C(X)$, we can take the functionals in Theorem 1 to be point-evaluations.

THEOREM 2. *Let* X *be a compact, Hausdorff space and let* L_n *be an n-dimensional subspace of* $C(X)$. *Then the following statements are equivalent*:

(i) *there is a positive projection of* $C(X)$ (*or of a closed linear sublattice of* $C(X)$) *onto* L_n,

(ii) *there exist non-negative fuctions* $b_1,..., b_n$ *in* L_n *and points* $x_1,..., x_n$ *in* X *such that* $b_i(x_j) = \delta_{ij}$.

Proof. Let the functionals f_i be as in Theorem 1, and let $S(f_i)$ denote the support of f_i. For $j \geqslant 2$, we have $f_1(b_j) = 0$, hence $b_j(x) = 0$ for all x in $S(f_1)$.

Since $f_1(b_1)=1$, there exists x_1 in $S(f_1)$ with $b_1(x_1)>0$. Replace b_1 by a positive scalar multiple to obtain $b_1(x_1)=1$. The points $x_2,\ldots,x_n$ are found similarly. ∎

Remarks. (1) In the same way, one sees that if for a certain set of non-negative functions $b_1,\ldots,b_n$ in L_n, there exist *unique* points x_i such that $b_i(x_i)>0$ and $b_i(x_j)=0$ for $j\neq i$, then there is a unique positive projection onto L_n.

(2) For any positive linear mapping T of $C(X)$ into itself, we have $\|T\|=\|Te\|$, where e is the function with constant value 1. Hence if P is a positive projection onto a subspace containing e, then $\|P\|=1$ (and if e is in the subspace, $\|P\|=1$ implies P is positive). For the positive projection $Pu=\sum_1^n u(x_i)\, b_i$ given by Theorem 2, we have $\|P\|=\|b_1+\cdots+b_n\|$. If the subspace does not contain e, this may well be greater than 1, even when a non-positive projection of norm 1 exists. To obtain a simple example, let X be a 3-point set, so that $C(X)$ is $\mathbb{R}^3$. Let L_2 be the subspace consisting of elements (x,y,z) satisfying $x=y+2z$. Then there is an unique positive projection given by $P(x,y,z)=(y+2z,y,z)$. (This corresponds to $b_1=(1,1,0)$, $b_2=(2,0,1)$.) Clearly, $\|P\|=3$. However, there is a non-positive projection with norm 1, namely, $Q(x,y,z)=(x,y,(x-y)/2)$.

On the other hand, if the subspace does contain e, then the problem of finding the minimal norm projection is equivalent, in a certain sense, to that of finding the "least negative" projection. By this we mean the following. It is easy to show that for $f\geqslant 0$,

$$\frac{1}{\|f\|}\min(Pf)(x)\geqslant\frac{1}{2}(1-\|P\|),$$

and equality is attained when we take the infimum over all $f\geqslant 0$.

It was shown by Morris and Cheney [2, Theorem 9] that if $n\geqslant 3$ and L_n is an n-dimensional Chebyshev subspace of $C[a,b]$ containing the constant functions, then every projection onto L_n has norm greater than 1. Consequently there is no positive projection onto L_n. Using our Theorem 2, we can prove the following stronger statement.

COROLLARY. *Let L_n be an n-dimensional subspace of $C[a,b]$. Assume that L_n contains an m-dimensional Chebyshev subspace X_m, where $m\geqslant 3$. Then there is no positive projection onto L_n.*

Proof. Since X_m is a Chebyshev subspace, for each x_0 in (a,b) there exists an f_0 in X_m such that $f_0\geqslant 0$ on $[a,b]$ and $f_0(x)=0$ only for $x=x_0$.

Assume that there is a positive projection of $C[a,b]$ onto L_n. Let $b_1,\ldots,b_n$ and $x_1,\ldots,x_n$ be as in the statement of Theorem 2. Choose x_0 in $(a,b)\backslash\{x_1,\ldots,x_n\}$, and let f_0 be as above. Express f_0 in the form $\sum_1^n\lambda_i b_i$.

Since $f_0(x_i) > 0$, we have $\lambda_i > 0$ for $i = 1,..., n$. However, $f_0(x_0) = 0$, so $b_i(x_0) = 0$ for each i. This holds for all x_0 as above, which implies that each b_i is identically zero, a contradiction. ∎

Actually, an even stronger statement is true. We say that a subspace E of $C(X)$ has the "Korovkin property" if the identity is the only positive operator of $C(X)$ into itself that agrees with the identity on E. (This differs slightly from the usual definition, which refers to a sequence of positive operators.) The Korovkin property implies, of course, that there is no positive projection onto E.

In the situation of the above corollary, X_m (and hence L_n) has the Korovkin property. For f in $C[a, b]$ and each x, set

$$\bar{f}(x) = \inf\{ g(x): g \in X_m, g \geqslant f \},$$
$$\underline{f}(x) = \sup\{ g(x): g \in X_m, g \leqslant f \}.$$

It is easily proved that if X_m is an m-dimensional Chebyshev subspace, with $m \geqslant 3$, then $\bar{f}(x) = \underline{f}(x) = f(x)$ for each x in (a, b). Let T be any positive operator of $C[a, b]$ into itself which is the identity on X_m, and take any f in $C[a, b]$. Then it follows from the above equalities that $(Tf)(x) = f(x)$ for all x in (a, b), and hence that $Tf = f$ (cf. Berens and Lorentz [1] and Šaškin [3]).

While the Korovkin property implies the non-existence of a positive projection, the converse is certainly not true. For example, if E has the Korovkin property, then so has any subspace containing E. No such inclusion property holds for the existence or non-existence of positive projections. Two specific examples will be given after Proposition 3.

When L_n is considered as a subspace of $L^p[a, b]$, a statement similar to the above corollary holds even for $m = 2$. In fact, we have as a corollary of Theorem 1:

COROLLARY. *Let L_n be an n-dimensional subspace of $C[a, b]$. Assume that L_n contains an m-dimensional Chebyshev subspace X_m, where $m \geqslant 2$. Then there is no positive projection of $L^p[a, b]$ onto L_n (where $1 \leqslant p < \infty$).*

Proof. By the previous corollary, we need only consider the case $m = 2$. Since X_2 is a Chebyshev subspace, there exist g in X_2 such that $g(a) = 0$ and $g(x) > 0$ for $a < x \leqslant b$, and h in X_2 such that $h(a) > 0$. Let $b_1,..., b_n$ and $f_1,..., f_n$ be as in the statement of Theorem 1. Then g can be expressed as $\sum_1^n \mu_i b_i$, with each $\mu_i \geqslant 0$. Since $g - \varepsilon h$ takes negative values for each $\varepsilon > 0$, we must have $\mu_k = 0$ for some k. It follows that $f_k(g) = 0$. But this is impossible, since f_k is a non-zero, non-negative element of $L^{p'}[a, b]$ and g is a non-negative, continuous function that vanishes only at a. ∎

Let π_n denote the space of algebraic polynomials of degree $\leqslant n$. It follows from the elementary form of Korovkin's theorem (or from the above) that there is no positive projection of $C[a, b]$ onto π_n for $n \geqslant 2$. We give here a simple direct proof of a slightly stronger statement, together with a variant which is not obtainable from Korovkin-type theorems. Let $C_0[a, b]$ denote the set of functions in $C[a, b]$ that are 0 at a, and let $\pi_n^0 = \pi_n \cap C_0[a, b]$.

PROPOSITION 3. (i) *If $n \geqslant 2$, then there is no positive projection of π_{n+1} onto π_n.*

(ii) *If $n \geqslant 3$, then there is no positive projection of π_{n+1}^0 onto π_n^0.*

Proof. It is sufficient to consider $[a, b] = [0, 1]$. Write $r_k(x) = x^k$. For both (i) and (ii), suppose that there is a positive projection P, and let $P(r_{n+1}) = u$. Now $0 \leqslant r_{n+1} \leqslant r_n$, so $0 \leqslant u \leqslant r_n$. It is elementary that this, together with the fact that u is in π_n, implies that $u = \alpha r_n$ for some α in $[0, 1]$. Now $x^{n+1} \geqslant nx^2 - (n-1)x$ for x in $[0, 1]$. Hence $u(x) \geqslant nx^2 - (n-1)x$. In particular, $u(1) \geqslant 1$, so $\alpha \geqslant 1$.

For (i), let

$$h_k(x) = k\left(x - \frac{1}{4}\right)^2 + \frac{1}{2.4^n}.$$

Then h_k is in π_2, and for a suitable k we have $h_k \geqslant r_{n+1}$ (h_k is a "narrow" quadratic having twice the value of r_{n+1} at $\frac{1}{4}$). Hence $h_k \geqslant u$. Evaluation at $\frac{1}{4}$ gives $\alpha \leqslant \frac{1}{2}$, a contradiction.

For (ii), modify this slightly, as follows. Let

$$g_k(x) = k\left(x - \frac{1}{4}\right)^2 + \frac{1}{2.4^{n-1}}.$$

Choose k such that $x^n \leqslant g_k(x)$, so that $x^{n+1} \leqslant x g_k(x)$. The function $x g_k(x)$ is in π_3^0, so we obtain $\alpha x^n \leqslant x g_k(x)$. This gives $\alpha \leqslant \frac{1}{2}$, as before. ∎

Remarks. (1) There is a positive projection of $C[0, 1]$ onto π_1, and in fact onto any two-dimensional subspace containing the constant functions. For if f is a non-constant function, then we can define $f_1 = \alpha f + \beta$ such that $0 \leqslant f_1 \leqslant 1$ and f_1 attains the values 0, 1. Then f_1 and $1 - f_1$ satisfy the conditions of Theorem 2. The positive projection onto π_1 is unique.

(2) The subspace π_n^0 does not have the Korovkin property in $C[0, 1]$. This is shown by the mapping $(Tf)(x) = f(x) + f(0)$.

An additional example, containing the constant functions, is as follows. Let L_3 denote the subspace of $C[-1, 1]$ spanned by the functions $1, x^2, x^4$. The mapping $(Tf)(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$ shows that L_3 does not have the

Korovkin property. By an application of Theorem 2 (and zero counting), or by the method of Proposition 3, it is easily seen that there is no positive projection onto L_3.

EXAMPLE. The subspace π_2^0, consisting of polynomials of the form $ax + bx^2$, displays several interesting features. Note first that $ax + bx^2 \geqslant 0$ on $[0, 1]$ if and only if a, $a + b \geqslant 0$. Consequently, π_2^0 has a basis $\{b_1, b_2\}$ such that $\lambda_1 b_1 + \lambda_2 b_2 \geqslant 0$ if and only if $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$: let $b_1(x) = x - x^2$, $b_2(x) = x^2$. We prove that there is no positive projection of $C[0, 1]$ onto π_2^0, showing that the existence of such a basis is not in itself sufficient. Let g_1, g_2 be the positive functionals on π_2^0 such that $g_i(b_j) = \delta_{ij}$. We show that there is no positive extension of g_1 defined on $C[0, 1]$ (or even on π_2); this implies the non-existence of a positive projection. Suppose, in fact, that f_1 were such an extension, and let $\alpha > 0$. Since

$$0 \leqslant (x - \alpha)^2 = x^2 - 2\alpha x + \alpha^2,$$

we have

$$0 \leqslant -2\alpha + \alpha^2 f_1(e),$$

so $f_1(e) \geqslant 2/\alpha$ for all $\alpha > 0$, which is impossible.

Since $C_0[0, 1]$ is a sublattice of $C[0, 1]$, it follows from the corollary of Theorem 1 that there is no positive projection of $C_0[0, 1]$ onto π_2^0. However, it is not hard to show that for each $n \geqslant 3$, there is a unique positive projection $P_n$of π_n^0 onto π_2^0: if $f(x) = a_1 x + \cdots + a_n x^n$, then $(P_n f)(x) = a_1 x + (a_2 + \cdots + a_n) x^2$. By considering $f_n(x) = 1 - (1 - x)^n$, one sees that $\|P_n\| \to \infty$ as $n \to \infty$.

3. FINITE-CODIMENSIONAL SUBSPACES OF $C(X)$

In this section, we prove:

THEOREM 4. *Let X be a compact, Hausdorff space with no isolated points. Then there is no positive projection of $C(X)$ onto any proper, finite-codimensional subspace.*

The proof will be achieved by a series of lemmas. Suppose that P is such a projection, and let E be its (finite-dimensional) kernel. We shall work with E rather the range of P. Note first that if $f \in E$ and $0 \leqslant g \leqslant f$, then $Pg = 0$, so $g \in E$ (that is, E is "order-convex"). We deduce:

LEMMA 1. *E contains no non-zero, non-negative function.*

Proof. Suppose that E contains a non-zero, non-negative function f. We may assume that $f \geqslant 1$ on some open set G. Since X has no isolated points, G is infinite. Let (x_n) be a sequence of distinct points in G. For each n, there is a function f_n in $C(X)$ such that $0 \leqslant f_n \leqslant 1$, $f_n(x_n) = 1$, $f_n(x_i) = 0$ for $i < n$ and $f_n(x) = 0$ for all x in $X \backslash G$. Then $0 \leqslant f_n \leqslant f$, so $f_n \in E$, and it is clear that the sequence (f_n) is linearly independent, contradicting the fact that E is finite-dimensional. ∎

LEMMA 2. *Let* $E_1 = \{f \in E: \|f\| = 1\}$, *and for* f *in* $C(X)$, *let* $i(f) = \inf\{f(x): x \in X\}$. *Then there exists* $c > 0$ *such that* $i(f) \leqslant -c$ *for all* $f \in E_1$.

Proof. This is clear, since i is continuous, E_1 is compact and, by Lemma 1, $i(f) < 0$ for $f \in E_1$. ∎

LEMMA 3. *There exist* $x_1, \ldots, x_k$ *in* X *such that if* $f \in E$ *and* $f(x_i) \geqslant 0$ *for each* i, *then* $f = 0$.

Proof. Let c be as in Lemma 2. Since E_1 is a compact subset of $C(X)$, it is equicontinuous. Therefore for each x in X, there is a neighbourhood $U(x)$ such that if $y \in U(x)$, then $|f(y) - f(x)| \leqslant c/2$ for all f in E_1. The space X can be covered by a finite choice of such neighbourhoods, say $U(x_1), \ldots, U(x_k)$. If f is in E_1, then $f(x) \leqslant -c$ for some x, so $f(x_i) \leqslant -c/2$ for some i. ∎

LEMMA 4. *Write* $Q = I - P$. *If* f *is in* $C(X)$ *and* $f(x_i) = 0$ *for all* i, *then* $Qf = 0$.

Proof. It is sufficient to prove this for non-negative f (then consider f^+ and f^-). Suppose that $f \geqslant 0$ and $f(x_i) = 0$ for each i. Then $Pf = f - Qf \geqslant 0$, so $(Qf)(x_i) \leqslant 0$ for all i. Since Qf is in E, Lemma 3 gives $Qf = 0$. ∎

Proof of Theorem 4. Choose some $f \geqslant 0$ with $Qf \neq 0$. By Lemma 1, Qf has both positive and negative values. Hence $(Qf)(y) > 0$ for some y different from $x_1, \ldots, x_k$. Let h be a non-negative function taking the value 1 at each x_i and 0 at y. Let $g = fh$. Then $g(x_i) = f(x_i)$ for all i, so by Lemma 4, $Qg = Qf$. In particular, $(Qg)(y) > 0$. But $g(y) = 0$, so $(Pg)(y) < 0$. This contradicts the positivity of P since $g \geqslant 0$. ∎

Remarks. (1) Let A be a closed, proper subset of X and let $C(X, A)$ denote the set of functions in $C(X)$ that vanish on A. The same reasoning shows that there is no positive projection of $C(X, A)$ onto any proper, finite-codimensional subspace of itself. It is well-known that every closed, order-convex linear sublattice (i.e., closed *lattice ideal*) of $C(X)$ is of the form $C(X, A)$.

(2) Let $C(X, x_0)$ denote the set of functions in $C(X)$ that vanish at x_0. If x_0 is not an isolated point (whether or not other isolated points exist), then there is no positive projection of $C(X)$ onto $C(X, x_0)$. This follows easily from the fact that any projection onto $C(X, x_0)$ has the form $Pf = f - f(x_0)\, g$, where g is a function with $g(x_0) = 1$.

4. Certain Subspaces of $C(X \times Y)$

We start with an elementary result.

Proposition 5. *Let X be a compact, Hausdorff space and Y any topological space. Let L_n be an n-dimensional subspace of $C(X)$ spanned by $f_1,\ldots,f_n$. Let B^i $(i = 1,\ldots, n)$ be subspaces of $C(Y)$, each containing the constant functions. Let A_n be the set of functions of the form $\sum_1^n f_i(x)\, g_i(y)$, where $g_i \in B^i$ for each i. If there is no positive projection of $C(X)$ onto L_n, then there is no positive projection of $C(X \times Y)$ onto A_n.*

Proof. Choose any $y^* \in Y$, and let $(Q\phi)(x) = \phi(x, y^*)$ for ϕ in $C(X \times Y)$. If P were a positive projection of $C(X \times Y)$ onto A_n, then $QP|_{C(X)}$ would be a positive projection of $C(X)$ onto L_n. ∎

A natural application of this is the extension of Proposition 3 to polynomials in two or more variables. For $x, y \in I = [0, 1]$, set

$$\tilde{\pi}_n^2 = \left\{ \sum a_{ij} x^i y^j : i + j \leqslant n \right\},$$

$$\pi_{n,m}^2 = \left\{ \sum a_{ij} x^i y^j : i \leqslant n, j \leqslant m \right\}.$$

Then for $n \geqslant 2$, there is no positive projection of $C(I^2)$ onto $\tilde{\pi}_n^2$ or $\pi_{n,m}^2$. (In fact, examination of the proofs of Propositions 3 and 5 shows easily that there is no positive projection of $\tilde{\pi}_{n+1}^2$ onto $\tilde{\pi}_n^2$, or of $\pi_{n+1,m}^2$ onto $\pi_{n,m}^2$.) There is a unique positive projection of $C(I^2)$ onto $\pi_{1,1}^2$, given by interpolation at the corners of I^2. It follows from Theorem 2 that there is no positive projection of $C(I^2)$ onto $\tilde{\pi}_1^2$. The method of proof of Theorem 6 actually shows that there is no positive projection of $\pi_{1,1}^2$ onto $\tilde{\pi}_1^2$.

Let X, Y be compact, Hausdorff spaces neither containing only one point. Let M be the subspace of $C(X \times Y)$ consisting of functions of the form $\phi(x, y) = f(x) + g(y)$. Also, for a chosen $x^* \in X$, $y^* \in Y$, let $C_0(X \times Y)$ denote the set of functions in $C(X \times Y)$ which vanish at (x^*, y^*), and let $M_0 = M \cap C_0(X \times Y)$. We shall consider both positive and minimal projections of $C(X \times Y)$ onto M, and of $C_0(X \times Y)$ onto M_0. Our first result totally characterizes the positive projections.

THEOREM 6. *There is no positive projection of* $C(X \times Y)$ *onto* M. *There is exactly one positive projection* P^* *of* $C_0(X \times Y)$ *onto* M_0, *given by* $(P^*\phi)(x, y) = \phi(x, y^*) + \phi(x^*, y)$.

Proof. Suppose that P is a positive projection of $C(X \times Y)$ onto M. Let $f \in C(X)$, $g \in C(Y)$ be arbitrary functions such that $0 \leqslant f, g \leqslant 1$, and for which there exist points $x_0, x_1 \in X$ and $y_0, y_1 \in Y$ satisfying $f(x_0) = g(y_0) = 0$, $f(x_1) = g(y_1) = 1$. Set $u(x, y) = f(x)g(y)$, and let $(Pu)(x, y) = h(x) + k(y)$, with $h(x_0) = 0$. Then $(Pu)(x_0, y) = k(y) \geqslant 0$ for all y in Y. Now, $f(x) \geqslant u(x, y)$, so $f(x) \geqslant h(x) + k(y)$ for all x, y. Taking $x = x_0$, we see that $k = 0$. It follows by similar reasoning that $h = 0$. By construction, $[1 - f(x)][1 - g(y)] \geqslant 0$. Application of P gives $1 - f(x) - g(y) \geqslant 0$ for all x, y. Set $x = x_1$ and $y = y_1$ to obtain a contradiction.

Obviously P^* is a positive projection of $C_0(X \times Y)$ onto M_0. Let P be any positive projection of $C_0(X \times Y)$ onto M_0. For $\phi \in C_0(X \times Y)$, define $\psi(x, y) = \phi(x, y) - \phi(x, y^*) - \phi(x^*, y)$. The result follows if we prove that $P\psi = 0$. Now, ψ enjoys the property that $\psi(x, y^*) = \psi(x^*, y) = 0$ for all x, y. Set $(P\psi)(x, y) = h(x) + k(y)$, where $h(x^*) = k(y^*) = 0$. Define $\psi^+(x) = \max\{\psi(x, y): y \in Y\}$, and $\psi^-(x) = \min\{\psi(x, y): y \in Y\}$. Thus $\psi^+(x) \geqslant \psi(x, y) \geqslant \psi^-(x)$ for all x, y. Since $\psi^+, \psi^- \in M_0$, it follows that $0 = \psi^+(x^*) \geqslant h(x^*) + k(y) \geqslant \psi^-(x^*) = 0$ for all $y \in Y$. Thus $k = 0$. Similarly $h = 0$. This proves the theorem. ∎

Remark. The first part of the proof also shows that there is no positive projection of $C(X \times Y)$ onto M_0. This is interesting in the light of the results of Sections 2 and 3.

For any $(x^*, y^*) \in X \times Y$ the map defined by $(P\phi)(x, y) = \phi(x, y^*) + \phi(x^*, y) - \phi(x^*, y^*)$ is a projection of $C(X \times Y)$ onto M of norm 3. We prove that this is minimal if both X and Y contain an infinite number of points.

THEOREM 7. *Let* X *and* Y *be infinite, compact, Hausdorff spaces. Let* P *be any projection of* $C(X \times Y)$ *onto* M. *Then* $\|P\| \geqslant 3$.

Proof. Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be any two sets of n distinct points in X and Y, respectively. Let $f_i \in C(X)$, $i = 1,..., n$, satisfy

$$f_i(x_j) = \delta_{ij}, \qquad i, j = 1,..., n,$$

$$f_i(x) \geqslant 0, \qquad i = 1,..., n, \quad x \in X,$$

$$\sum_{i=1}^{n} f_i(x) = 1, \qquad x \in X.$$

Similarly define $g_i \in C(Y)$, $i = 1,..., n$, with respect to the points $\{y_i\}_{i=1}^n$.

Set $\phi^{rs}(x, y) = f_r(x)\, g_s(y)$, $r, s = 1,..., n$. It easily follows that

$$\phi^{rs}(x_i, y_j) = \delta_{ri}\delta_{sj}, \qquad r, s, i, j = 1,..., n, \tag{1}$$

$$\sum_{r=1}^{n} \phi^{rs}(x, y) = g_s(y), \qquad s = 1,..., n, \tag{2}$$

$$\sum_{s=1}^{n} \phi^{rs}(x, y) = f_r(x), \qquad r = 1,..., n. \tag{3}$$

Now, set $(P\phi^{rs})(x_i, y_j) = a_{ij}^{rs}$.

From (1), (2), and (3), we obtain

$$\sum_{r=1}^{n} a_{ij}^{rs} = \delta_{sj}, \qquad s, i, j = 1,..., n, \tag{4}$$

$$\sum_{s=1}^{n} a_{ij}^{rs} = \delta_{ri}, \qquad r, i, j = 1,..., n. \tag{5}$$

Furthermore, since P is a projection onto M, and functions in M satisfy $\phi(z_0, w_0) + \phi(z_1, w_1) = \phi(z_0, w_1) + \phi(z_1, w_0)$, we have

$$a_{ij}^{rs} = a_{i1}^{rs} + a_{1j}^{rs} - a_{11}^{rs}, \qquad r, s, i, j = 1,..., n. \tag{6}$$

By (6),

$$\sum_{i,j=1}^{n} a_{ij}^{ij} = \sum_{i,j=1}^{n} [a_{i1}^{ij} + a_{1j}^{ij} - a_{11}^{ij}],$$

and applying (4) and (5) we obtain

$$\sum_{i,j=1}^{n} a_{ij}^{ij} = 2n - 1.$$

We show that $a_{ij}^{ij} \geqslant (3 - \|P\|)/2$ for each i, j. It then follows that

$$2n - 1 \geqslant (3 - \|P\|)\, n^2/2$$

or $\|P\| \geqslant 3 - (4n - 2)/n^2$. This is true for all n, so $\|P\| \geqslant 3$.

Consider $i, j = 1$ (a similar proof holds for each choice of i, j). Since $0 \leqslant (1 - f_1(x))(1 - g_1(y)) \leqslant 1$, we have

$$|1 - 2f_1(x) - 2g_1(y) + 2f_1(x)\, g_1(y)| \leqslant 1$$

for all $x \in X$, $y \in Y$. Applying P and evaluating at $x = x_1$, $y = y_1$, gives

$$|1 - 2f_1(x_1) - 2g_1(y_1) + 2(P\phi^{11})(x_1, y_1)| \leqslant \|P\|$$

from which $|-3 + 2a_{11}^{11}| \leqslant \|P\|$. Thus $a_{11}^{11} \geqslant (3 - \|P\|)/2$. ∎

Remark. If X contains exactly n points and Y contains exactly m points, then the above argument shows that for any projection P of $C(X \times Y)$ onto M, $\|P\| \geqslant 3 - (2n + 2m - 2)/nm$. This lower bound is in fact attained by the choice

$$\begin{aligned} a_{ij}^{rs} &= -1/nm, & r \neq i, \quad s \neq j, \\ &= (n-1)/nm, & r = i, \quad s \neq j, \\ &= (m-1)/nm, & r \neq i, \quad s = j, \\ &= (n+m-1)/nm, & r = i, \quad s = j, \end{aligned}$$

where the $\{a_{ij}^{rs}\}_{r,i=1}^{n}{}_{s,j=1}^{m}$ are understood to be defined as in the proof of Theorem 7.

The projection P^* of $C_0(X \times Y)$ onto M_0 as given in Theorem 6 is of norm 2. Our next result, which is a variant of Theorem 7, shows that this is minimal.

THEOREM 8. *Let X and Y be infinite, compact, Hausdorff spaces. Let P be any projection of $C_0(X \times Y)$ onto M_0. Then $\|P\| \geqslant 2$.*

Proof. As in the proof of Theorem 7, let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be any two sets of n distinct points ($n \geqslant 2$) in X and Y, respectively, with $x_1 = x^*$ and $y_1 = y^*$. Let $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^n$ be as in the proof of Theorem 7, and set $\phi^{rs}(x, y) = f_r(x) g_s(y)$ for $r, s = 1, \ldots, n$; $(r, s) \neq (1. 1)$. For notational convenience, set $\phi^{11} = 0$. Thus $\phi^{rs} \in C_0(X \times Y)$ for all $r, s = 1, \ldots, n$. Now

$$\phi^{rs}(x_i, y_j) = \delta_{ri}\delta_{sj}, \qquad r, s, i, j = 1, \ldots, n, \quad (r, s) \neq (1, 1), \tag{7}$$

$$\sum_{r=1}^{n} \phi^{rs}(x, y) = g_s(y), \qquad s = 2, \ldots, n, \tag{8}$$

$$\sum_{s=1}^{n} \phi^{rs}(x, y) = f_r(x), \qquad r = 2, \ldots, n. \tag{9}$$

Set $(P\phi^{rs})(x_i, y_j) = a_{ij}^{rs}$. Since P maps $C_0(X \times Y)$ onto M_0, $a_{11}^{rs} = 0$, $r, s = 1, \ldots, n$. From the definition of a_{ij}^{rs}, we have

$$a_{ij}^{rs} = a_{i1}^{rs} + a_{1j}^{rs}, \qquad a_{ij}^{11} = 0, \qquad a_{11}^{rs} = 0, \qquad r, s, i, j = 1, \ldots, n, \tag{10}$$

and from (7), (8) and (9),

$$\sum_{r=1}^{n} a_{ij}^{rs} = \delta_{sj}, \qquad s = 2,..., n, \tag{11}$$

$$\sum_{s=1}^{n} a_{ij}^{rs} = \delta_{ri}, \qquad r = 2,..., n. \tag{12}$$

By (10), (11) and (12),

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}^{ij} &= \sum_{i,j=1}^{n} [a_{i1}^{ij} + a_{1j}^{ij}] \\ &= \sum_{i=2}^{n} \sum_{j=1}^{n} a_{i1}^{ij} + \sum_{j=2}^{n} \sum_{i=1}^{n} a_{1j}^{ij} \\ &= 2(n-1).\end{aligned}$$

We show that $a_{ij}^{ij} \geqslant 2 - \|P\|$, for $i, j = 1,..., n$, $(i, j) \neq (1, 1)$. It then follows that

$$2(n-1) = \sum_{i,j=1}^{n} a_{ij}^{ij} \geqslant (n^2 - 1)(2 - \|P\|)$$

or $\|P\| \geqslant 2 - 2/(n+1)$. This is true for all n, so $\|P\| \geqslant 2$.

We divide the proof of this claim into two cases. First assume that $i, j > 1$. For convenience, consider $i = j = n$. Since $0 \leqslant f_n, g_n \leqslant 1$,

$$0 \leqslant f_n(x) + g_n(y) - f_n(x)\, g_n(y) \leqslant 1.$$

Because $f_n + g_n - f_n g_n \in C_0(X \times Y)$, we can apply P and evaluate at $x = x_n$, $y = y_n$ to obtain

$$|2 - a_{nn}^{nn}| \leqslant \|P\|.$$

So $a_{nn}^{nn} \geqslant 2 - \|P\|$.

Now assume that either i or j, but not both, is equal to 1. For convenience set $i = 1$, $j = n$. The function $g_n - (1 - f_1)(1 - g_n) \in C_0(X \times Y)$, and since $0 \leqslant f_1, g_n \leqslant 1$,

$$|\, g_n(y) - (1 - f_1(x))(1 - g_n(y))| \leqslant 1.$$

Applying P and evaluating at $x = x_1$, $y = y_n$ gives

$$|2 - a_{1n}^{1n}| \leqslant \|P\|.$$

So $a_{1n}^{1n} \geqslant 2 - \|P\|$. This completes the proof. ∎

Remark. Theorems 7 and 8 generalize as follows. Let $X^1, X^2,..., X^k$ be infinite, compact, Hausdorff spaces. There exists no positive projection of $C(X^1 \times X^2 \times \cdots \times X^k)$ onto $M = C(X^1) + C(X^2) + \cdots + C(X^k)$, and any projection of $C(X^1 \times X^2 \times \cdots \times X^k)$ onto M is of norm at least $2k-1$. Let $x_i^* \in X^i$, $i = 1,..., k$. Set $C_0(X^1 \times \cdots \times X^k) = \{\phi : \phi \in C(X^1 \times \cdots \times X^k), \phi(x_1^*,..., x_k^*) = 0\}$ and $M_0 = M \cap C_0(X^1 \times \cdots \times X^k)$. There is a unique positive projection P^* of $C_0(X^1 \times \cdots \times X^k)$ onto M_0, given by $(P^*\phi)(x_1,..., x_k) = \phi(x_1, x_2^*,..., x_k^*) + \cdots + \phi(x_1^*, x_2^*,..., x_k)$. For every projection P of $C_0(X^1 \times \cdots \times X^k)$ onto M_0, $\|P\| \geqslant \|P^*\| = k$.

If X^i contains exactly m_i points, $i = 1,..., k$, then every projection P of $C(X^1 \times \cdots \times X^k)$ onto M satisfies

$$\|P\| \geqslant (2k-1) - (2k-2)\left[\sum_i 1/m_i - \sum_{i \neq j} 1/m_i m_j + \cdots + (-1)^{k-1}/m_1 m_2 \cdots m_k\right],$$

and this lower bound is attained (see the remark after Theorem 7).

Acknowledgment

We are indebted to Dr. W. A. Light for his interest in this work.

Note added in proof. Remark. Many results relating to positive projections are to be found in Donner [5]. In particular, it follows as a special case of his Theorem 4.7 that a finite-dimensional subspace L_n of a Banach lattice L admits a positive projection if (and only if) it (i) is a lattice in the induced ordering and (ii) any subset of L_n that has an upper bound in L has an upper bound in L_n. Our Theorem 1 provides a simple proof of this. One need only establish that any positive linear functional on L_n has a positive extension defined on L. This is an immediate consequence of the Hahn–Banach theorem and the fact that there is a K such that $\|(x^+)_{L_n}\| \leqslant K \|(x^+)_L\|$ for all $x \in L_n$, where $(x^+)_E = \sup_E(x, 0)$. To prove this, assume instead that there are elements x_n with $\|(x_n^+)_L\| \leqslant 2^{-n}$ and $\|(x_n^+)_{L_n}\| > n$. Let $y = \sum_1^\infty (x_n^+)_L$ and $A = \{x \in L_n : x \leqslant y\}$. Then A contains 0 and all x_n, which leads to a contradiction of (ii).

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Printed in Belgium